Measure Theory with Ergodic Horizons
Lecture 6

Null and measurable sets.

Def. A *measure space* is a triple $(X, \mathcal{B}, \mu)$, where $\mathcal{B}$ is a σ-alg on the set X and μ is a measure on $\mathcal{B}$. The measure space $(X, \mathcal{B}, \mu)$ is called *standard* if X is a Polish space, $\mathcal{B} := \mathcal{B}(X)$, and μ is σ-finite.

Def. Given a measure space $(X, \mathcal{B}, \mu)$, a set $Z \subseteq X$ is called *μ-null* if $Z \subseteq \tilde{Z}$ where $\tilde{Z} \in \mathcal{B}$ and $\mu(\tilde{Z}) = 0$. Denote by *Null_μ* the collection of all μ-null sets.
For sets A, B, we write $A =_\mu B$ if $A \Delta B$ is μ-null.
Call a set $M \subseteq X$ *μ-measurable* if $M =_\mu B$ for some $B \in \mathcal{B}$. Denote by *Meas_μ* the collection of all μ-measurable sets. Equiv., $M = B \Delta Z$ where $Z \in \mathrm{Null}_\mu$.

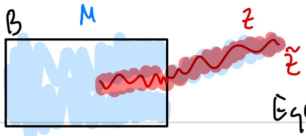
Prop. Let $(X, \mathcal{B}, \mu)$ be a measure space.
(a) Null_μ is a *σ-ideal*, i.e. is closed downward under $\subseteq$ and under ctbl unions.
(b) Meas_μ is a σ-algebra, in fact, $\mathrm{Meas}_\mu = \langle \mathcal{B} \cup \mathrm{Null}_\mu \rangle_\sigma$.

Proof. (a) If (Z_n) are null sets, then $\exists\, \tilde{Z}_n \in \mathcal{B}$ with $\tilde{Z}_n \supseteq Z_n$ and $\mu(\tilde{Z}_n) = 0$, so
$$\bigcup_{n\in\mathbb{N}} Z_n \subseteq \bigcup_{n\in\mathbb{N}} \tilde{Z}_n \in \mathcal{B} \text{ and } \mu\Big(\bigcup_{n\in\mathbb{N}} \tilde{Z}_n\Big) \leq \sum_{n\in\mathbb{N}} \mu(\tilde{Z}_n) = 0.$$

(b) The closure under complements follows from: $M =_\mu B \iff M^c =_\mu B^c$ (since $M \Delta B = M^c \Delta B^c$).
For ctbl unions, let $M_n =_\mu B_n \in \mathcal{B}$. Then $\bigcup_{n\in\mathbb{N}} M_n =_\mu \bigcup_{n\in\mathbb{N}} B_n$ because $\Big(\bigcup_{n\in\mathbb{N}} M_n\Big) \Delta \Big(\bigcup_{n\in\mathbb{N}} B_n\Big) \subseteq \bigcup_{n\in\mathbb{N}} (M_n \Delta B_n)$, which is null by (a). $\square$

Prop. Let $(X, \mathcal{B}, \mu)$ be a measure space.
$\{B \cup Z : B \in \mathcal{B} \text{ and } Z \in \mathrm{Null}_\mu\} = \mathrm{Meas}_\mu = \{B \setminus Z : B \in \mathcal{B} \text{ and } Z \in \mathrm{Null}_\mu\}$.

Proof. Enough to show $\supseteq \mathrm{Meas}_\mu \subseteq$. Let $M \in \mathrm{Meas}_\mu$ so $M =_\mu B \in \mathcal{B}$, i.e. $Z := M \Delta B$ is null.



Equivalently, $M = B \triangle Z$. Let $\tilde{Z} \supseteq Z$ be a null set in $\mathcal{B}$ and let $B' := B \setminus \tilde{Z}$, $\tilde{B} := B \cup \tilde{Z}$. Then $M = B' \cup ((B \cap \tilde{Z}) \setminus Z) \cup (Z \setminus B)$ and $M = \tilde{B} \setminus ((B \cap Z) \cup ((\tilde{Z} \setminus B) \setminus Z))$. $\square$

Remark. It is a HW exercise to check that for a finite premeasure on $\mathcal{A}$, the σ-algebras $\mathcal{M}$ in both Carathéodory's and Tao's proofs are exactly $\{B \triangle Z : B \in \langle \mathcal{A} \rangle_\sigma$ and Z is μ-null$\}$.

Def. A measure space $(X, \mathcal{B}, \mu)$ is called *complete* if $\mathcal{B} = \mathrm{Meas}_\mu$.

Prop (Completion). Every measure space $(X, \mathcal{B}, \mu)$ admits a unique completion, i.e. there is a unique extension of μ to a measure $\bar{\mu}$ on Meas_μ.

Proof. For existence, for any $M \in \mathrm{Meas}_\mu$, write it as $B \cup Z$ and define $\bar{\mu}(M) := \mu(B)$. We need to show that this definition is independent of the choice of B. Indeed, if $B_1 \cup Z_1 = B_2 \cup Z_2$, where $B_i \in \mathcal{B}$ and Z_i is μ-null, then

$$\mu(B_1) = \mu(B_1 \cup \tilde{Z}_1 \cup \hat{Z}_2) \geq \mu(B_2),$$ hence also $\mu(B_1) \leq \mu(B_2)$ by symmetry.

For σ-additivity, let (M_n) be pairwise disjoint μ-measurable sets and let $M_n = B_n \sqcup Z_n$ where $B_n \in \mathcal{B}$ and $Z_n \in \mathrm{Null}_\mu$. Then

$$\bar{\mu}\Big(\bigsqcup_n M_n\Big) = \bar{\mu}\Big(\bigsqcup_n B_n \sqcup \bigsqcup_n Z_n\Big) := \mu\Big(\bigsqcup_n B_n\Big)$$ because $\bigsqcup_n B_n \in \mathcal{B}$ and $\bigsqcup_n Z_n$ is null.

Then $\mu\Big(\bigsqcup_n B_n\Big) = \sum_n \mu(B_n) =: \sum_n \bar{\mu}(M_n)$.

For uniqueness, if ν was any other extension to a measure on Meas_μ, then for each $M \in \mathrm{Meas}_\mu$, let $M = B \sqcup Z$ where $B \in \mathcal{B}$ and $Z \in \mathrm{Null}_\mu$. Hence

$$\mu(B) = \nu(B) \leq \nu(M) \leq \nu(B \sqcup \tilde{Z}) = \nu(B) + \nu(\tilde{Z}) = \mu(B) + \mu(\tilde{Z}) = \mu(B)$$

so $\nu(M) = \mu(B) = \mu(M)$, where $\tilde{Z} \supseteq Z$ is a null set from $\mathcal{B}$. $\square$

In the future, given any measure space $(X, \mathcal{B}, \mu)$, we will always be working with its completion $\bar{\mu}$ and still denote it by μ. In particular, we will write $\mu(M) = 0.7$ for a set $M \in \mathrm{Meas}_\mu$ that's in $\mathcal{B}$.

Remark on cardinalities of $\mathcal{B}(X)$ vs. Meas_μ. Let $(X, \mathcal{B}, \mu)$ be a standard measure space with $|X| =$ continuum.

(a) There are continuum many open sets: let $\{U_n\}_{n\in\mathbb{N}}$ be a ctbl basis. Then each open set U can be encoded by $x_U \in 2^{\mathbb{N}}$, where

$$x_U(n) := \begin{cases} 1 & \text{if } U_n \subseteq U \\ 0 & \text{o.w.} \end{cases}$$

Then $U \mapsto x_U$ is an injection of the collection of open sets into $2^{\mathbb{N}}$.

(b) There are continuum many Borel sets:
let Σ^0_1 denote the collection of open sets. For each ctbl ordinal α, if Σ_α is defined, then we define

$$\Pi^0_\alpha := \neg \Sigma_\alpha := \{B^c : B \in \Sigma_\alpha\}.$$

And if Σ_β is defined for all $\beta < \alpha$, then

$$\Sigma_\alpha := \Big\{ \bigcup_{n\in\mathbb{N}} B_n : B_n \in \Pi^0_{\beta_n} \text{ for } \beta_n < \alpha \Big\}.$$

Also let $\Delta^0_n := \Sigma^0_n \cap \Pi^0_n$. Then

$$\underbrace{\begin{array}{ccccccccccccccc} & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & & & \Sigma^0_\alpha & & \\ \Delta^0_1 & \subsetneq & & \subsetneq & \Delta^0_2 & \subsetneq & & \subsetneq & \cdots & \cdots & \Delta^0_\alpha & \subsetneq & & \subsetneq & \Delta^0_{\alpha+1} \subsetneq \cdots \\ & & \Pi^0_1 & \underset{\text{HW1}}{\subsetneq} & & & \Pi^0_2 & & & \omega & & & \Pi^0_\alpha & & \qquad \omega_1 \end{array}}_{\mathcal{B}(X)}$$

$$\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Delta^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha.$$

One shows by induction on $\alpha < \omega_1$ that each Σ^0_α has size $\leq$ continuum, hence $\mathcal{B}(X)$ has size $\leq$ continuum because continuum $\times\, \omega_1 =$ continuum.

(c) $|\mathrm{Meas}_\mu| = 2^{\text{continuum}} > \text{continuum}.$

This is because there are null sets Z of size continuum, e.g. the standard Cantor set in $[0,1]$, and all subsets of Z are μ-null hence μ-measurable, i.e. $\mathcal{P}(Z) \subseteq \mathrm{Meas}_\mu$, so $|\mathrm{Meas}_\mu| \geq 2^{|Z|} = 2^{\text{continuum}}$.